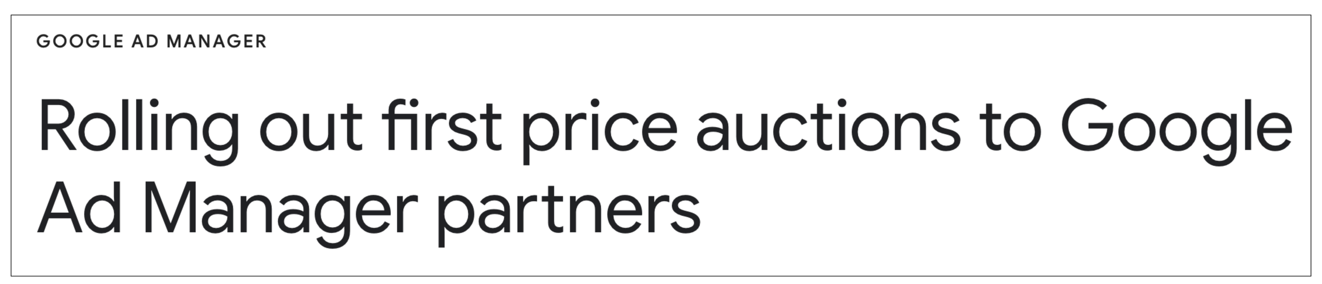
Contextual Standard Auctions with Budgets

Santiago Balseiro, Christian Kroer, Rachitesh Kumar

Columbia University

Motivation



- **First-Price Auctions:** Major ad platforms have recently shifted to first-price auctions as the preferred auction format.
- **Contextual Values:** Users have features which determine the value of advertisers.
- **Budgets:** Buyers often have caps on average/total expenditure.

Model

- **Setup:** Seller (ad platform) auctions one indivisible item among n buyers (advertisers).
- **Feature-based Values:** Item type represented by feature vector $\alpha \in A \subset \mathbb{R}^d$, buyer type represented by weight vector and budget $(w, B) \in \Theta \subset \mathbb{R}^d$. Value is given by $w^T\alpha$.
- **Bayesian Setting:** Buyer types are drawn from commonly known distribution G, item types are drawn from commonly known distribution F, buyer types are private information, item type is revealed before auction

Symmetric First Price Equilibrium

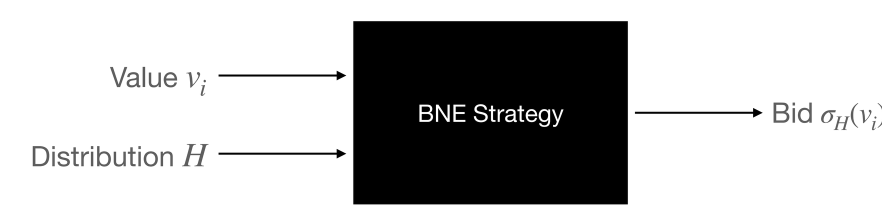
A strategy $\beta^* : \Theta \times A \to \mathbb{R}_{\geq 0}$ such that $\beta^*(w, B, \alpha)$ is an optimal solution to the following optimization problem a.s. $(w, B) \sim G$:

$$\max_{b:A\to\mathbb{R}_{\geq 0}} \quad \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[(w^T\alpha - b(\alpha))\, 1\{b(\alpha) \geq \{\beta^*(\theta_i, \alpha)\}_i\}\right]$$
$$\text{s.t.} \quad \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[b(\alpha)\, 1\{b(\alpha) \geq \{\beta^*(\theta_i, \alpha)\}_i\}\right] \leq B.$$

Pacing Based SFPE

- **IID, No Budgets, No Contexts:**
  - Values of all buyers are i.i.d. $v_i \sim H$
  - Well-known: Symmetric equilibrium exists and is given by

$$\sigma_H(v_i) = E\left[\max_{j\neq i} v_i \;\middle|\; \max_{j\neq i} v_i < v\right]$$



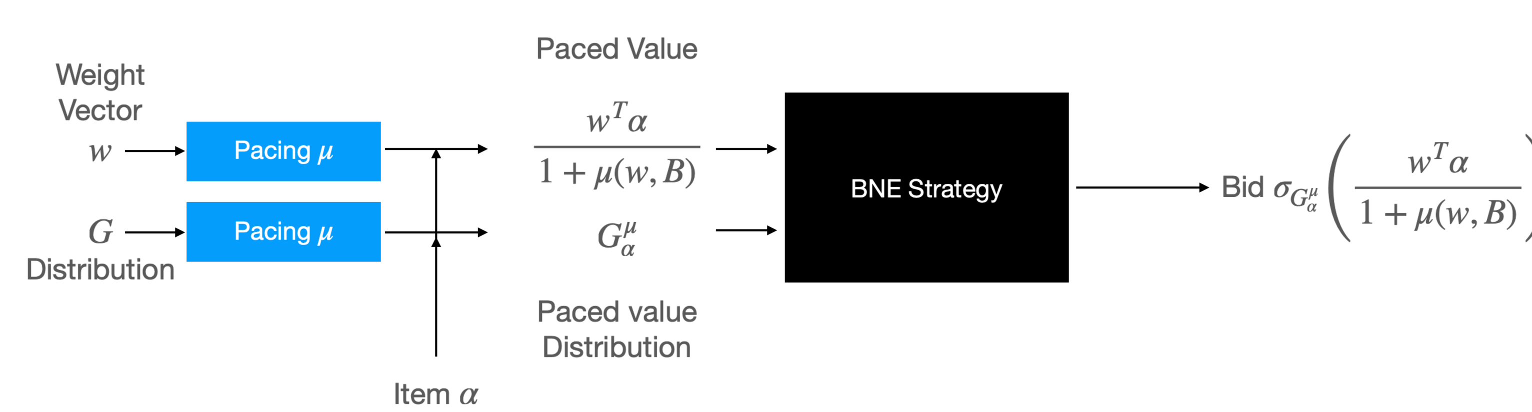
- **Pacing:** Each buyer type (w, B) has an associated dual variable $\mu(w, B)$ corresponding to the budget constraint in her optimization problem, her paced value for item α is given by

$$\frac{w^T\alpha}{1 + \mu(w, B)}$$

- **Pacing Based Strategy:** Each buyer treats her paced value as her true value, assumes that every other buyer does the same, and bids according to σ_α

$$\beta(w, B, \alpha) = \sigma_{G^\mu_\alpha}\left(\frac{w^T\alpha}{1 + \mu(w, B)}\right)$$

- **Theorem:** There exists pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that the pacing based strategy corresponding to μ is a SFPE.



- **Novel Proof Technique:** We use a fixed point theorem in the infinite-dimensional space of dual multipliers μ by exploiting topological properties of multi-variable functions of bounded variation.

Standard Auctions and Revenue Equivalence

- **Theorem:** For anonymous standard auctions other than first-price auctions: the BNE strategy **(black box)** changes, but the same equilibrium pacing function **(blue box)** remains the same, leading to revenue equivalence. **This is surprising because revenue equivalence does not hold for strict budget constraints.**

Price of Anarchy of Liquid Welfare

- **Liquid Welfare:** Maximum revenue that can be extracted from the buyers by an omniscient seller with complete knowledge of their values.
- **Theorem:** Value-pacing-based equilibria of all standard auctions have a Price of Anarchy $\geq 1/2$ for liquid welfare.

Numerical Simulations

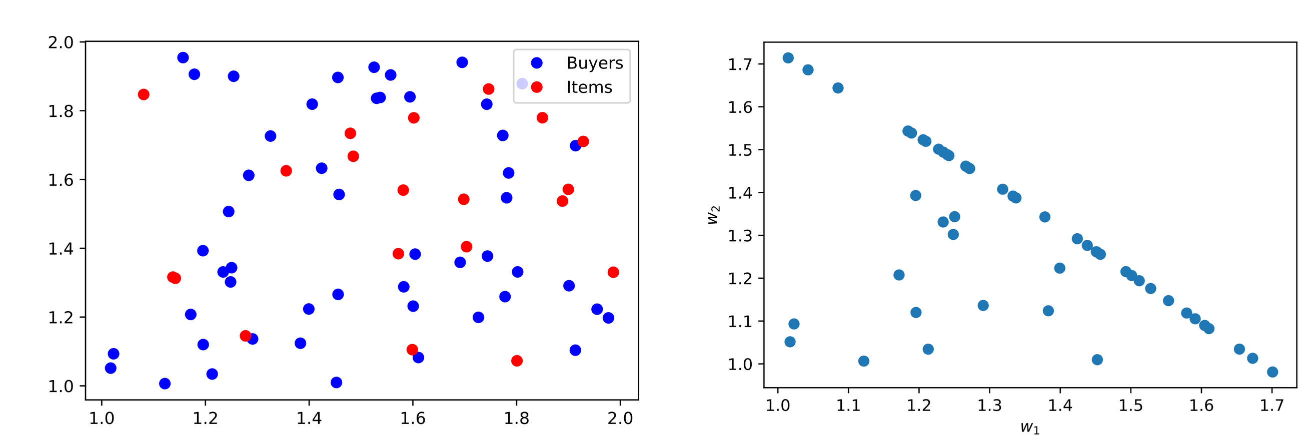


Figure 1:The first plot depicts the set of possible buyer and item types. 3 buyers participate in the auction, each sampling her weight vector uniformly from the blue points, and the item is sampled uniformly from the red points. All buyers have budget B = 2. The second plot shows the paced weight vectors of the buyers in the SFPE, which was computed through best-response dynamics in the dual space. **In our structural results, we explain why weight vectors get paced down to the same level, as seen in the above plots.**