The Stationary Prophet Inequality Problem

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Model: Stationary Prophet Inequality

- Infinite horizon two-sided market
- Homogenous Items supplied at random times (Poisson process with arrival rate λ), & are **perishable** (at exponential rate 1)
- Buyers of type $i \in [n]$ arrive at random times (via independent Poisson processes with arrival rate γ_i), bid a fixed value v_i for any item, & are **impatient** (depart immediately)
- Upon a buyer's arrival, must immediately accept/reject bid
- **Objective:** maximize expected average revenue

Classic Prophet Inequality

$X_1 \sim D_1$ $X_2 \sim D_2$ … $X_n \sim D_n$

- n treasure chests, initially locked, are opened one by one
- When a chest is opened, its prize $X_i \sim D_i$ is revealed and a **gambler** must immediately & irrevocably accept or reject
- **Q:** How well does gambler compare to omniscient **prophet**?

knows realizations $x_1, \dots, x_n$

prophet $= E\left[\max_i X_i\right]$

knows distributions $D_1, \dots, D_n$

gambler $= ?$

- **Thm:** gambler's optimal policy ≥ 1/2 · prophet (tight) [1]
- **Thm:** gambler's posted price policy ≥ 1/2 · prophet [2]

Benchmarks

Optimal Offline (OPT_{off})

omnipotent & omniscient

- Classic PI: $E[\max_i X_i]$
- Stationary PI: E[max-weight matching]

Optimal Online (OPT_{on})

omnipotent but **not** omniscient

- Classic PI: finite-dim. DP
- Stationary PI: infinite-dim. DP

Our Question

- How well do **"simple"** policies approximate optimal ones?
- Simple = **single-price policy** (v, p):
  accept bid > v, accept bid = v with prob. p, reject any bid < v

Our Results

- **Thm 1:** There exists a single-price policy that is **1/2** competitive with OPT_{off}. No online policy can do better.
- **Thm 2:** There exists a single-price policy that is a **0.656**-approximation of OPT_{on}.

	previous best known	our results
vs OPT_{off}	1/3 [3]	1/2 (optimal)
vs OPT_{on}	1 – 1/e ≈ 0.632 [4]	0.656

- Our results also hold subject to **scarce inventory space**.

LP Benchmark (OPT_{off})

- **Idea:** Design LP capturing constraints for optimal algorithm's marginal sale probabilities s.t.

$$LP_{off} \geq OPT_{off}$$

$$(LP_{off}) \quad \max \sum_i v_i \cdot x_i$$

$$\text{s.t.} \sum_i x_i \leq \lambda \qquad (1)$$

$$x_i \leq \gamma_i \cdot (1 - \exp(-\lambda)) \quad \forall i \qquad (2)$$

$$x_i \geq 0 \quad \forall i$$

- **Variables:** $x_i/\gamma_i = \Pr[\text{type } i \text{ buyer is sold an item by } OPT_{off}]$
- **Constraint (1):** conservation of mass
- **Constraint (2):**

$$\Pr[\text{type } i \text{ buyer sold item by } OPT_{off}] \leq \Pr[\text{at least one item is available}]$$

Bound RHS using **PASTA** property [5]

Algorithm & Analysis

- Design s.t. $\Pr[\text{type } i \text{ buyer sold an item}] \geq 1/2 \cdot x_i/\gamma_i \ \forall i$
- Approximation follows by linearity of expectation

Algorithm

for arrival of buyer of type i

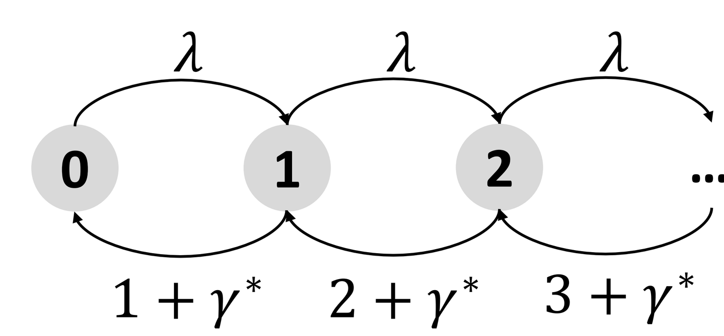
if ≥ 1 items available

sell to buyer w.p. $p_i := \frac{x_i}{\gamma_i \cdot (1-\exp(-\lambda))}$

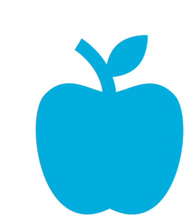
Analysis outline

$$\Pr[\text{type } i \text{ buyer sold an item}] = p_i \cdot \Pr[\geq 1 \text{ item available}] = \frac{x_i}{\gamma_i} \cdot \underbrace{\frac{\Pr[\geq 1 \text{ item available}]}{1 - \exp(-\lambda)}}_{\geq 1/2}$$

(main technical contribution)



Proof of Optimality

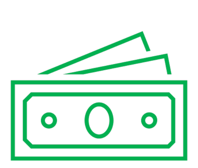


Items

$\lambda = \epsilon$

scarce supply

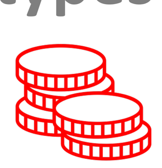
Two buyer types



$\gamma_1 = \epsilon$

$v_1 = 1 + 1/\epsilon$

rare big spender



$\gamma_2 = \infty$

$v_2 = 1$

common miser

Analysis outline:

$$\lim_{\epsilon \to 0} \frac{OPT_{on}(\epsilon)}{OPT_{off}(\epsilon)} = \frac{1}{2}$$

References

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